The Euler equations from the point of view of differential inclusions

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Abstract: We show that for a Schrödinger operator with bounded potential on a manifold with cylindrical ends, the space of solutions that grows at most exponentially at infinity is finite dimensional and, for a dense set of potentials (or, equivalently, for a surface for a fixed potential and a dense set of metrics), the constant function 0 is the only solution that vanishes at infinity. Clearly, for general potentials there can be many solutions that vanish at infinity. One of the key ingredients in these results is a three circles inequality (or log convexity inequality) for the Sobolev norm of a solution u to a Schrödinger equation on a product N × [0, T], where N is a closed manifold with a certain spectral gap. Examples of such N's are all (round) spheres n for n 1 and all Zoll surfaces. Finally, we discuss some examples arising in geometry of such manifolds and Schrödinger operators.

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THE EULER EQUATIONS AS A DIFFERENTIAL INCLUSION

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Abstract. In this paper we propose a new point of view on weak solutions of the Euler equations, describing the motion of an ideal incompressible fluid in $\mathbb{R}^n$ with $n \geq 2$. We give a reformulation of the Euler equations as a differential inclusion, and in this way we obtain transparent proofs of several celebrated results of V. Scheffer and A. Shnirelman concerning the non-uniqueness of weak solutions and the existence of energy–decreasing solutions. Our results are stronger because they work in any dimension and yield bounded velocity and pressure.

1. Introduction

Consider the Euler equations in n space dimensions, describing the motion of an ideal incompressible fluid,

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - f &= 0 \\ \operatorname{div} v &= 0\,. \end{aligned} \tag{1}$$

Classical (i.e. sufficiently smooth) solutions of the Cauchy problem exist locally in time for sufficiently regular initial data and driving forces (see Chapter 3.2 in [16]). In two dimensions such existence results are available also for global solutions (e.g. Chapters 3.3 and 8.2 in [16] and the references therein). Classical solutions of Euler's equations with $f = 0$ conserve the energy, that is $t \mapsto \int |v(t,x)|^2\,dx$ is a constant function. Hence the energy space for (1) is $L^\infty_t(L^2_x)$.

A recurrent issue in the modern theory of PDEs is that one needs to go beyond classical solutions, in particular down to the energy space (see for instance [6, 8, 16, 23]). A divergence–free vector field $v \in L^2_{loc}$ is a *weak solution* of (1) if

$$\int v\partial_t\varphi + \langle v \otimes v, \nabla\varphi\rangle - v \cdot f\,dx\,dt \;=\; 0 \tag{2}$$

for every test function $\varphi \in C^\infty_c(\mathbb{R}^n_x \times \mathbb{R}_t, \mathbb{R}^n)$ with $\operatorname{div}\varphi = 0$. It is well–known that then the pressure is determined up to a function depending only on time (see [26]). In the case of Euler strong motivation for considering weak solutions comes also from mathematical physics, especially the theory of turbulence laid down by Kolmogorov in 1941 [3, 11]. A celebrated criterion of Onsager related to Kolmogorov's theory says, roughly speaking, that dissipative weak solutions cannot have a Hölder exponent greater than 1/3 (see [4, 9, 10, 18]). It is therefore of interest to construct weak solutions with limited regularity.

Weak solutions are not unique. In a well–known paper [20] Scheffer constructed a surprising example of a weak solution to (1) with compact support in space and time when $f = 0$ and $n = 2$. Scheffer's proof is very long and complicated and a simpler construction was later given by Shnirelman in [21]. However, Shnirelman's proof is still quite difficult. In this paper we obtain a short and elementary proof of the following theorem.

Theorem 1.1. *Let $f = 0$. There exists $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$ and $p \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving (1) in the sense of distributions, such that v is not identically zero, and $\operatorname{supp} v$ and $\operatorname{supp} p$ are compact in space-time $\mathbb{R}^n_x \times \mathbb{R}_t$.*

In mathematical physics weak solutions to the Euler equations that dissipate energy underlie the Kolmogorov theory of turbulence. In another groundbreaking paper [22] Shnirelman proved the existence of L^2 distributional solutions with $f = 0$ and energy which decreases in time. His methods are completely unrelated to those in [20] and [21]. In contrast, the following extension of his existence theorem is a simple corollary of our construction.

Theorem 1.2. *There exists (v, p) as in Theorem 1.1 such that, in addition:*
- *$\int |v(t, x)|^2\, dx = 1$ for every $t \in]-1, 1[$;*
- *$v(t, x) = 0$ for $|t| > 1$.*

Our method has several interesting features. First of all, our approach fits nicely in the well–known framework of L. Tartar for the analysis of oscillations in linear partial differential systems coupled with nonlinear pointwise constraints [7, 15, 24, 25]. Roughly speaking, Tartar's framework amounts to a plane–wave analysis localized in physical space, in contrast with Shnirelman's method in [21], which is based rather on a wave analysis in Fourier space. In combination with Gromov's convex integration or with Baire category arguments, Tartar's approach leads to a well understood mechanism for generating irregular oscillatory solutions to differential inclusions (see [14, 15, 17]).

Secondly, the velocity field we construct belongs to the energy space $L^\infty_t(L^2_x)$. This was not the case for the solutions in [20, 21], and it was a natural question whether weak solutions in the energy space were unique. Our first theorem shows that even higher summability assumptions of v do not rule out such pathologies. The pressure in [20, 21] is only a distribution solving (1). In our construction p is actually the potential–theoretic solution of

$$-\Delta p \;=\; \partial^2_{x_i x_j}(v^i v^j) - \partial_{x_i} f_i\,. \tag{3}$$

However, being bounded, it has slightly better regularity than the BMO given by the classical estimates for (3).

Next, our point of view reveals connections between the apparently unrelated constructions of Scheffer and Shnirelman. Shnirelman considers sequences of driving forces f_k converging to 0 in some negative Sobolev space.

In particular he shows that for a suitable choice of f_k the corresponding solutions of (1) converge in L^2 to a nonzero solution of (1) with $f = 0$. Scheffer builds his solution by iterating a certain piecewise constant construction at small scales. On the one hand both our proof and Scheffer's proof are based on oscillations localized in physical space. On the other hand, our proof gives as an easy byproduct the following approximation result in Shnirelman's spirit.

Theorem 1.3. *All the solutions* (v, p) *constructed in the proofs of Theorem 1.1 and in Theorem 1.2 have the following property. There exist three sequences* $\{v_k\}, \{f_k\}, \{p_k\} \subset C_c^\infty$ *solving* (1) *such that*

- f_k *converges to* 0 *in* H^{-1}*;*
- $\|v_k\|_\infty + \|p_k\|_\infty$ *is uniformly bounded;*
- $(v_k, p_k) \to (v, p)$ *in* L^q *for every* $q < \infty$.

Our results give interesting information on which kind of additional (entropy) condition could restore uniqueness of solutions. As already remarked, belonging to the energy space is not sufficient. In fact, in view of our method of construction, there is strong evidence that neither energy–decreasing nor energy–preserving solutions are unique. In a forthcoming paper we plan to investigate this issue, and also the class of initial data for which our method yields energy–decreasing solutions.

The rest of the paper is organized as follows. In Section 2 we carry out the plane wave analysis of the Euler equations in the spirit of Tartar, and we formulate the core of our construction (Proposition 2.2). In Section 3 we prove Proposition 2.2. In Section 4 we show how our main results follow from the Proposition. We emphasize that the concluding argument in Section 4 appeals to the – by now standard – Baire category methods for solving differential inclusions [1, 2, 5, 13], and in particular our main source is the work of B. Kirchheim [14]. We point out that for this part of the proof we could have used convex integration as pioneered by S. Müller and V. Šverák [17] in connection with regularity in elliptic systems. In fact we believe that for $n \geq 3$ an approach closer in spirit to Gromov's original one (see [12]) would also work, yielding solutions which are even continuous (work in progress). However, we chose to present the Baire category argument, because for the purposes of this paper it is the shortest and most elegant.

2. Plane wave analysis of Euler's equations

We start by briefly explaining Tartar's framework [24]. One considers nonlinear PDEs that can be expressed as a system of linear PDEs (conservation laws)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{4}$$

coupled with a pointwise nonlinear constraint (constitutive relations)

$$z(x) \in K \subset \mathbb{R}^d \text{ a.e.}, \tag{5}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable. The idea is then to consider *plane wave* solutions to (4), that is, solutions of the form

$$z(x) = ah(x \cdot \xi), \tag{6}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* Λ is given by the states $a \in \mathbb{R}^d$ such that for any choice of the profile h the function (6) solves (4), that is,

$$\Lambda := \left\{ a \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \setminus \{0\} \quad \sum_{i=1}^m \xi_i A_i a = 0 \right\}. \tag{7}$$

Oscillatory behavior of solutions to the nonlinear problem is then determined by the compatibility of the set K with the cone Λ.

The Euler equations can be naturally rewritten in this framework. The domain is $\mathbb{R}^m = \mathbb{R}^{n+1}$, and the state variable z is defined as $z = (v, u, q)$, where

$$q = p + \frac{1}{n}|v|^2, \text{ and } u = v \otimes v - \frac{1}{n}|v|^2 I_n,$$

so that u is a symmetric $n \times n$ matrix with vanishing trace and I_n denotes the $n \times n$ identity matrix. From now on the linear space of symmetric $n \times n$ matrices will be denoted by $\mathcal{S}^n$ and the subspace of trace–free symmetric matrices by $\mathcal{S}_0^n$. The following lemma is straightforward.

Lemma 2.1. *Suppose $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$, $u \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathcal{S}^n_0)$, and $q \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ such that*

$$\begin{aligned} \partial_t v + div\ u + \nabla q &= 0, \\ div\ v &= 0, \end{aligned} \tag{8}$$

in the sense of distributions. Then v and p defined by $p := q - \frac{1}{n}|v|^2$ are a solution to (1) with $f = 0$ if and only if

$$u = v \otimes v - \frac{1}{n}|v|^2 I_n \qquad a.e.\ in\ \mathbb{R}^n_x \times \mathbb{R}_t. \tag{9}$$

Consider the $(n+1) \times (n+1)$ symmetric matrix in block form

$$U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}, \tag{10}$$

where I_n is the $n \times n$ identity matrix. Notice that by introducing new coordinates $y = (x, t) \in \mathbb{R}^{n+1}$ the equation (8) becomes simply

$$\mathrm{div}_y U = 0.$$

Here, as usual, a divergence–free matrix field is a matrix of functions with rows that are divergence–free vectors. Therefore the wave cone corresponding to (8) is given by

$$\Lambda = \left\{ (v, u, q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R} : \det \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} = 0 \right\}.$$

Remark 1. *A simple linear algebra computation shows that for every $v \in \mathbb{R}^n$ and $u \in \mathcal{S}_0^n$ there exists $q \in \mathbb{R}$ such that $(v, u, q) \in \Lambda$, revealing that the wave cone is very large. Indeed, let $V^\perp \subset \mathbb{R}^n$ be the linear space orthogonal to v and consider on $V^\perp$ the quadratic form $\xi \mapsto \xi \cdot u\xi$. Then, $\det(U) = 0$ if and only if $-q$ is an eigenvalue of this quadratic form.*

In order to exploit this fact for constructing irregular solutions to the nonlinear system, one needs plane wave–like solutions to (8) which are localized in space. Clearly an exact plane–wave as in (6) has compact support only if it is identically zero. Therefore this can only be done by introducing an error in the range of the wave, deviating from the line spanned by the wave state $a \in \mathbb{R}^d$. However, this error can be made arbitrarily small. This is the content of the following proposition, which is the building block of our construction.

Proposition 2.2 (Localized plane waves)**.** *Let $a = (v_0, u_0, q_0) \in \Lambda$ with $v_0 \neq 0$, and denote by σ the line segment in $\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}$ joining the points $-a$ and a. Then there exists a constant $\alpha > 0$ such that for every $\varepsilon > 0$ there exists a smooth solution (v, u, q) of (8) with the properties*

- *the support of (v, u, q) is contained in $B_1(0) \subset \mathbb{R}^n_x \times \mathbb{R}_t$;*
- *the image of (v, u, q) is contained in the ε–neighborhood of σ;*
- *$\int |v(x,t)|\, dx\, dt \geq \alpha$.*

3. Localized plane waves

For the proof of Proposition 2.2 there are two main points. Firstly, we appeal to a particular large group of symmetries of the equations in order to reduce the problem to some special Λ-directions. Secondly, to achieve a cut-off which preserves the linear equations (8), we introduce a suitable potential.

Definition 3.1. *We denote by $\mathcal{M}$ the set of symmetric $(n+1) \times (n+1)$ matrices A such that $A_{(n+1)(n+1)} = 0$. Clearly, the map*

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (v, u, q) \quad \mapsto \quad U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} \in \mathcal{M} \tag{11}$$

is a linear isomorphism.

As already observed, in the variables $y = (x, t) \in \mathbb{R}^{n+1}$, the equation (8) is equivalent to $\operatorname{div} U = 0$. Therefore Proposition 2.2 follows immediately from

Proposition 3.2. *Let $\overline{U} \in \mathcal{M}$ be such that $\det \overline{U} = 0$ and $\overline{U} e_{n+1} \neq 0$, and consider the line segment σ with endpoints $-\overline{U}$ and $\overline{U}$. Then there exists a constant $\alpha > 0$ such that for any $\varepsilon > 0$ there exists a smooth divergence–free matrix field $U : \mathbb{R}^{n+1} \to \mathcal{M}$ with the properties*

(p1) $\operatorname{supp} U \subset B_1(0)$,
(p2) dist $(U(y), \sigma) < \varepsilon$ *for all* $y \in B_1(0)$,

(p3) $\int |U(y)e_{n+1}|dy \geq \alpha$.

The proof of Proposition 3.2 relies on two lemmas. The first deals with the symmetries of the equations.

Lemma 3.3 (The Galilean group)**.** *Let $\mathcal{G}$ be the subgroup of $GL_{n+1}(\mathbb{R})$ defined by*

$$\left\{A \in \mathbb{R}^{(n+1)\times(n+1)} : \det A \neq 0, Ae_{n+1} = e_{n+1}\right\}. \tag{12}$$

For every divergence–free map $U : \mathbb{R}^{n+1} \to \mathcal{M}$ and every $A \in \mathcal{G}$ the map

$$V(y) := A^t \cdot U(A^{-t}y) \cdot A$$

is also a divergence–free map $V : \mathbb{R}^{n+1} \to \mathcal{M}$.

The second deals with the potential.

Lemma 3.4 (Potential in the general case)**.** *Let $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{n+1})$ be functions for $i, j, k, l = 1, \dots, n+1$ so that the tensor E is skew–symmetric in ij and kl, that is*

$$E_{ij}^{kl} = -E_{ij}^{lk} = -E_{ji}^{kl} = E_{ji}^{lk}. \tag{13}$$

Then

$$U_{ij} = \mathcal{L}(E) = \frac{1}{2}\sum_{k,l} \partial_{kl}^2 (E_{kj}^{il} + E_{ki}^{jl}) \tag{14}$$

is symmetric and divergence–free. If in addition

$$E_{(n+1)i}^{(n+1)j} = 0 \qquad \text{for every } i \text{ and } j, \tag{15}$$

then U takes values in $\mathcal{M}$.

Remark 2. *A suitable potential in the case $n = 2$ can be obtained in a more direct way. Indeed, let $w \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be a divergence–free vector field and consider the map $U : \mathbb{R}^3 \to \mathcal{M}$ given by*

$$U \;=\; \begin{pmatrix} \partial_2 w_1 & \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 & \frac{1}{2}\partial_2 w_3 \\ \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 & -\partial_1 w_2 & -\frac{1}{2}\partial_1 w_3 \\ \frac{1}{2}\partial_2 w_3 & -\frac{1}{2}\partial_1 w_3 & 0 \end{pmatrix}. \tag{16}$$

Then it can be readily checked that U is divergence–free. Moreover, w is the curl of a vector field ω. However, this is just a particular case of Lemma 3.4. Indeed, given E as in the Lemma define the tensor $D_{ij}^k = \sum_l \partial_l E_{ij}^{kl}$. Note that D is skew–symmetric in ij and for each ij, the vector $(D_{ij}^k)_{k=1,\dots,n+1}$ is divergence–free. Moreover,

$$U_{ij} = \frac{1}{2}\sum_k \partial_k (D_{kj}^i + D_{ki}^j).$$

Then the vector field w above is simply the special choice where $D_{12}^k = -D_{21}^k = w_k$ and all other D's are zero, and a correspondent relation can be found for E and ω.

The proofs of the two Lemmas will be postponed until the end of the section and we now come to the proof of the Proposition.

Proof of Proposition 3.2. The proof consists of a reduction step followed by the direct construction of some special cases.

Step 1. We claim that it suffices to consider the particular case when $\overline{U}_{1j} = \overline{U}_{j1} = 0$ for every j, in other words $\overline{U}e_1 = 0$.

Take indeed a general $\overline{U}$ as in the statement of the Lemma. Since $\overline{U}$ has determinant equal to 0, its kernel is non–trivial. Let f be a nonzero element of the kernel and note that f is linearly independent from e_{n+1}. Let $f_1, \ldots, f_{n+1}$ be a basis for $\mathbb{R}^{n+1}$ such that $f_1 = f$ and $f_{n+1} = e_{n+1}$ and consider the matrix A such that $Ae_i = f_i$. Clearly A belongs to the group $\mathcal{G}$ defined in Lemma 3.3. Set

$$\overline{V} = A^t U A. \tag{17}$$

Clearly $\overline{V}$ has the property that $\overline{V}e_1 = 0$ and it belongs to $\mathcal{M}$.

So, let $\varepsilon > 0$. In Step 2 we will construct a smooth map V supported in $B_1(0)$ with the image lying in the ε–neighborhood of the line segment τ with endpoints $-\overline{V}$ and $\overline{V}$ and satisfying (p1), (p2) and (p3) of the proposition for some $\alpha > 0$.

We remark that A is invertible, and hence $A^{-t}(B_r(0)) \subset B_1(0)$ for some constant $r > 0$. By considering $V_r(y) = V(r^{-1}y)$ we obtain a corresponding map supported on $B_r(0)$. With some abuse of notation we continue to denote it by V.

Let U be the $\mathcal{M}$–valued map

$$U(x) = (A^{-1})^t V(A^t x) A^{-1}.$$

By the discussion above the linear isomorphism given by $X \mapsto (A^{-1})^t X A^{-1}$ maps the line segment τ onto σ. Therefore:

- U is supported in $B_1(0)$ and it is smooth;
- U is divergence–free thanks to Lemma 3.3;
- U takes values in a $C\varepsilon$–neighborhood of the segment σ;
- $\int_{B_1(0)} |Ue_3| \geq C' \int_{B_r(0)} |Ve_3|$.

Here the constants C and C' are positive and depend only on the matrix A, which in turns depends only on $\overline{U}$, but not on ε. This completes the proof of the second step.

Step 2. So far we have reduced our proposition to exhibiting a smooth divergence–free map $U : \mathbb{R}^{n+1} \to \mathcal{M}$ satisfying (p1), (p2) and (p3) in the special case where $\overline{U}e_1 = 0$.

In this case we simply set

$$E^{j1}_{i1} = -E^{j1}_{1i} = -E^{1j}_{i1} = E^{1j}_{1i} = \overline{U}_{ij} \frac{\sin(Ny_1)}{N^2}$$

and all the other entries equal to 0. Note that by our assumption $\overline{U}_{ij} = 0$ whenever one index is 1 or both of them are $n+1$. This ensures that the tensor E is well defined and satisfies the properties of Lemma 3.4.

In the case $n = 2$, the matrix takes necessarily the form

$$\overline{U} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & 0 \end{pmatrix} \tag{18}$$

and we can use the potential of Remark 2 by simply setting

$$w = \frac{1}{N}(0, a\cos(Ny_1), 2b\cos(Ny_1))$$

and

$$\omega = \frac{1}{N^2}(0, 2b\sin(Ny_1), -a\sin(Ny_1)).$$

Next, fix a smooth cutoff function φ such that

- $|\varphi| \leq 1$,
- $\varphi = 1$ on $B_{1/2}(0)$,
- $\mathrm{supp}\,(\varphi) \subset B_1(0)$,

and consider the map $U = \mathcal{L}(\varphi E)$. Clearly, U is smooth and supported in $B_1(0)$. By Lemma 3.4, U is divergence–free and $\mathcal{M}$–valued. Moreover

$$\int |U(y)e_{n+1}|dy \geq |\overline{U}e_{n+1}| \int_{B_{1/2}(0)} |\sin(Ny_1)|\, dy \geq \alpha,$$

for some positive constant $\alpha = \alpha(n)$ independent of N.

Next, observe that

$$U - \varphi\tilde{U} = \mathcal{L}(\varphi E) - \varphi\mathcal{L}(E)$$

is a sum of products of first–order derivatives of φ with first–order derivatives of components of E and of second–order derivatives of φ with components of E. Thus,

$$\|U - \varphi\tilde{U}\|_\infty \leq C\|\varphi\|_{C^2}\|E\|_{C^1} \leq \frac{C'}{N}\|\varphi\|_{C^2},$$

and by choosing N sufficiently large we obtain $\|U - \varphi\tilde{U}\|_\infty < \varepsilon$. On the other hand, since $|\varphi| \leq 1$ and $\tilde{U}$ takes values in σ, the image of $\varphi\tilde{U}$ is also contained in σ. This shows that the image of U is contained in the ε–neighborhood of σ and concludes the proof of Proposition 3.2. □

Proof of Lemma 3.3. First of all we check that whenever $B \in \mathcal{M}$, then $A^tBA \in \mathcal{M}$ for all $A \in \mathcal{G}$. Indeed, A^tBA is symmetric, and since A satisfies $Ae_{n+1} = e_{n+1}$, we have

$$\begin{aligned} (A^tBA)_{(n+1)(n+1)} &= e_{n+1} \cdot A^tBAe_{n+1} = Ae_{n+1} \cdot BAe_{n+1} \\ &= e_{n+1} \cdot Be_{n+1} = B_{(n+1)(n+1)} = 0. \end{aligned} \tag{19}$$

Now, let A, U and V be as in the statement. The argument above shows that V is $\mathcal{M}$–valued. It remains to check that if U is divergence–free, then V

is also divergence–free. To this end let $\phi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ be a compactly supported test function and consider $\tilde\phi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ defined by

$$\tilde\phi(x) = A\phi(A^t x).$$

Then $\nabla\tilde\phi(x) = A\nabla\phi(A^t x)A^t$, and by a change of variables we obtain

$$\begin{aligned}
\int \mathrm{tr}\big(\mathrm{V(y)}\nabla\phi(\mathrm{y})\big)\mathrm{dy} &= \int \mathrm{tr}\big(\mathrm{A^t U(A^{-t}y) A}\nabla\phi(\mathrm{y})\big)\mathrm{dy}\\
&= \int \mathrm{tr}\big(\mathrm{U(A^{-t}y)A}\nabla\phi(\mathrm{y})\mathrm{A^t}\big)\mathrm{dy}\\
&= \int \mathrm{tr}\big(\mathrm{U(x)A}\nabla\phi(\mathrm{A^t x})\mathrm{A^t}\big)(\det \mathrm{A})^{-1}\mathrm{dx}\\
&=(\det A)^{-1}\int \mathrm{tr}\big(\mathrm{U(x)}\nabla\tilde\phi(\mathrm{x})\big)\mathrm{dx} = 0,
\end{aligned}$$

since U is divergence–free. But this implies that V is also divergence-free. □

Proof of Lemma 3.4. First of all, U is clearly symmetric and $U_{(n+1)(n+1)} = 0$. Hence U takes values in $\mathcal{M}$. To see that U is divergence–free, we calculate

$$\begin{aligned}
\sum_j \partial_j U_{ij} &= \frac{1}{2}\sum_{k,l}\partial^3_{jkl}(E^{il}_{kj} + E^{jl}_{ki})\\
&= \frac{1}{2}\sum_l \partial_l\Big(\sum_{jk}\partial^2_{jk}E^{il}_{kj}\Big) + \frac{1}{2}\sum_k \partial_k\Big(\sum_{jl}\partial^2_{jl}E^{jl}_{ki}\Big) \overset{(13)}{=} 0\,.
\end{aligned}$$

This completes the proof of the lemma. □

4. Proof of the main results

For clarity we now state the precise form of our main result. Theorems 1.1, 1.2 and 1.3 are direct corollaries.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_t$ be a bounded open domain. There exists $(v,p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving the Euler equations*

$$\begin{aligned}
\partial_t v + \mathrm{div}\,(v\otimes v) + \nabla p &= 0\\
\mathrm{div}\, v &= 0\,,
\end{aligned}$$

such that

- *$|v(x,t)| = 1$ for a.e. $(x,t) \in \Omega$,*
- *$v(x,t) = 0$ and $p(x,t) = 0$ for a.e. $(x,t) \in (\mathbb{R}^n_x \times \mathbb{R}_t)\setminus\Omega$.*

Moreover, there exists a sequence of functions $(v_k, p_k, f_k) \in C_c^\infty(\Omega)$ such that

$$\begin{aligned}
\partial_t v_k + \mathrm{div}\,(v_k\otimes v_k) + \nabla p_k &= f_k\\
\mathrm{div}\, v_k &= 0\,,
\end{aligned}$$

and

- *f_k converges to 0 in H^{-1};*

- $\|v_k\|_\infty + \|p_k\|_\infty$ *is uniformly bounded;*
- $(v_k, p_k) \to (v, p)$ *in* L^q *for every* $q < \infty$*.*

We remark that the statements of Theorem 1.1 and Theorem 1.3 are just subsets of the statement of Theorem 4.1. As for Theorem 1.2, note that it suffices to choose, for instance, $\Omega = B_r(0) \times]-1,1[$, where $B_r(0)$ is the ball of $\mathbb{R}^n$ with volume 1.

We recall from Lemma 2.1 that for the first half of the theorem it suffices to prove that there exist

$$(v, u, q) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R})$$

with support in Ω, such that $|v| = 1$ a.e. in Ω and (8) and (9) are satisfied. In Proposition 2.2 we constructed compactly supported solutions (v, u, q) to (8). The point is thus to find solutions which satisfy in addition the pointwise constraint (9). The main idea is to consider the sets

$$K = \left\{ (v, u) \in \mathbb{R}^n \times \mathcal{S}^n_0 : \, u = v \otimes v - \frac{1}{n}|v|^2 I_n \, , \; |v| = 1 \right\} , \tag{20}$$

and

$$\mathcal{U} = \text{int } (K^{co} \times [-1, 1]), \tag{21}$$

where int denotes the topological interior of the set in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}$, and K^{co} denotes the convex hull of K. Thus, a triple (v, u, q) solving (8) and taking values in the convex extremal points of $\overline{\mathcal{U}}$ is indeed a solution to (9). We will prove that $0 \in \mathcal{U}$, and therefore there exist plane waves taking values in $\mathcal{U}$. The goal is to add them so to get an infinite sum

$$(v, u, q) = \sum_{i=1}^{\infty} (v_i, u_i, q_i)$$

with the properties that

- the partial sums $\sum_{i=0}^{k} (v_i, u_i, q_i)$ take values in $\mathcal{U}$,
- (v, u, q) is supported in Ω,
- (v, u, q) takes values in the convex extremal points of $\overline{U}$ a.e. in Ω,
- (v, u, q) solves the linear partial differential equations (8).

There are two important reasons why this construction is possible. First of all, since the wave cone Λ is very large, we can always get closer and closer to the extremal point of $\mathcal{U}$ with the sequence (v_k, u_k, p_k). Secondly, because the waves are localized in space–time, by choosing the supports smaller and smaller we can achieve strong convergence of the sequence. In view of Lemma 2.1 this gives the solution of Euler that we were looking for. The partial sums give the approximating sequence of the theorem.

This sketch of the proof is philosophically closer to the method of convex integration, where the difficulty is to ensure strong convergence of the partial sums. The Baire category argument avoids this difficulty by introducing a metric for the space of solutions to (8) with values in $\mathcal{U}$, and proving that in its closure a generic element takes values in the convex extreme points.

An interesting corollary of the Baire category argument is that, within the class of solutions to the Euler equations with driving force in some particular bounded subset of H^{-1}, the typical (in the sense of category) element has the properties of Theorem 4.1 .

Proof of Theorem 4.1. **Step 1 - the geometric setup**

Let K and $\mathcal{U}$ be defined as in (20) and (21), i.e.

$$K=\left\{(v,u)\in\mathbb{S}^{n-1}\times\mathcal{S}^n_0\,:\,u=v\otimes v-\frac{I_n}{n}\right\}.$$

We claim that $0\in\mathcal{U}$. This amounts to showing that $0\in\text{int }K^{co}$.

Let μ be the Haar measure on $\mathbb{S}^{n-1}$ and consider the linear map

$$T:C(\mathbb{S}^{n-1})\to\mathbb{R}^n\times\mathcal{S}^n_0,\quad \phi\mapsto\int_{\mathbb{S}^{n-1}}\left(v,v\otimes v-\frac{I_n}{n}\right)\phi\,d\mu\,.$$

Clearly, if

$$\phi\geq 0\qquad\text{and}\qquad\int_{\mathbb{S}^{n-1}}\phi\,d\mu=1\,, \tag{22}$$

then $T(\phi)\in K^{co}$. Notice that

$$T(1)\;=\;\int_{\mathbb{S}^{n-1}}\left(v,v\otimes v-\frac{I_n}{n}\right)d\mu\;=\;0,$$

and hence $0\in K^{co}$. Moreover, whenever $\psi\in C(\mathbb{S}^{n-1})$ is such that

$$\alpha\;=\;1-\int_{\mathbb{S}^{n-1}}\psi\,d\mu\;\geq\;\|\psi\|_{C(\mathbb{S}^{n-1})}, \tag{23}$$

$\phi=\alpha+\psi$ satisfies (22) and hence $T(\psi)=T(\phi)\in K^{co}$. Since (23) holds whenever $\|\psi\|_{C(\mathbb{S}^{n-1})}<1/2$, it suffices to show that T is surjective to prove that K^{co} contains a neighborhood of 0.

The surjectivity of T follows from orthogonality in $L^2(\mathbb{S}^{n-1})$. Indeed, letting $\phi=v_i$ for each i, we obtain

$$T(\phi)=\beta_1(e_i,0),\text{ where }\beta_1=\int_{\mathbb{S}^{n-1}}v_1^2d\mu.$$

Furthermore, setting $\phi=v_iv_j$ with $i\neq j$, we obtain

$$T(\phi)=\beta_2\big(0,e_i\otimes e_j+e_j\otimes e_i\big),\text{ where }\beta_2=\int_{\mathbb{S}^{n-1}}v_1^2v_2^2d\mu.$$

Finally, setting $\phi=v_i^2-\frac{1}{n}$ we obtain

$$T(\phi)=\beta_3\Big(0,e_i\otimes e_i-\frac{1}{(n-1)}\sum_{j\neq i}e_j\otimes e_j\Big),$$

where

$$\beta_3=\int_{\mathbb{S}^{n-1}}\left(v_1^2-\frac{1}{n}\right)^2d\mu.$$

This shows that the image of T contains $n+\frac{1}{2}n(n+1)-1$ linearly independent elements, hence a basis for $\mathbb{R}^n\times\mathcal{S}^n_0$.

Next, we claim that whenever $a,b\in\mathbb{S}^{n-1}$, the matrix

$$\begin{pmatrix} a\otimes a-\frac{I_n}{n} & a\\ a & 0\end{pmatrix}-\begin{pmatrix} b\otimes b-\frac{I_n}{n} & b\\ b & 0\end{pmatrix}$$

has zero determinant and hence in particular lies in the wave cone Λ defined in (7). Let $P\in GL_n(\mathbb{R})$ with $Pa=e_1$ and $Pb=e_2$. Note that

$$\begin{pmatrix} P & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} a\otimes a & a\\ a & 0\end{pmatrix}\begin{pmatrix} P^t & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix} Pa\otimes Pa & Pa\\ Pa & 1\end{pmatrix},$$

so that it suffices to check the determinant of

$$\begin{pmatrix} e_1\otimes e_1 & e_1\\ e_1 & 0\end{pmatrix}-\begin{pmatrix} e_2\otimes e_2 & e_2\\ e_2 & 0\end{pmatrix}.$$

Since $e_1+e_2-e_{n+1}$ is in the kernel of this matrix, it has indeed determinant zero.

Step 2 - the functional setup

We define the complete metric space X as follows. Let

$$X_0:=\left\{(v,u,q)\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}_t):\ \begin{array}{l}\bullet\,\mathrm{supp}\,(v,u,q)\subset\Omega\\ \bullet\,(v,u,q)\text{ solution of (8) in }\mathbb{R}^n_x\times\mathbb{R}_t\\ \bullet\,(v,u,q)\in\mathcal{U}\text{ for all }(x,t)\in\mathbb{R}^n_x\times\mathbb{R}_t\end{array}\right\}$$

equipped with the topology of L^∞-weak* convergence of (v,u,q). Let X be the closure of X_0 in this topology. In Step 1 we showed that the origin $(0,0,0)$ is contained in $\mathcal{U}$. Therefore the trivial triple $(v,u,q)=(0,0,0)$ belongs to X_0, and hence X is not empty. Moreover, by weak* convergence, any $(v,u,q)\in X$ satisfies

$$\mathrm{supp}\,(v,u,q)\subset\Omega,\quad (v,u,q)\text{ solves (8) and takes values in }\overline{\mathcal{U}}.$$

For the last assertion notice that $\overline{\mathcal{U}}$ is a compact and convex set.

Since X is a bounded subset of L^∞, the weak* topology is metrizable. We fix such a metric d^*_∞. Thus (X,d^*_∞) is a complete metric space.

Now consider the identity map

$$I:(X,d^*_\infty)\to L^1(\mathbb{R}^n_x\times\mathbb{R}_t)\ \text{ defined by }(v,u,q)\mapsto(v,u,q).$$

Let $\phi_r(x,t)=r^{-(n+1)}\phi(rx,rt)$ be any regular space-time convolution kernel. For each fixed $(v,u,q)\in X$ we have

$$(\phi_r*v,\phi_r*u,\phi_r*q)\to(v,u,q)\text{ strongly in }L^1\text{ as }r\to 0.$$

On the other hand, for each $r>0$ and $(v^k,u^k,q^k)\in X$

$$(v^k,u^k,q^k)\overset{*}{\rightharpoonup}(v,u,q)\text{ in }L^\infty\implies\phi_r*(v^k,u^k,q^k)\to\phi_r*(v,u,q)\text{ in }L^1.$$

Therefore each map $I_r:(X,d^*_\infty)\to L^1$ defined by

$$I_r:(u,v,q)\mapsto(\phi_r*v,\phi_r*u,\phi_r*q)$$

is continuous, and

$$I(v,u,q)=\lim_{r\to0}I_r(v,u,q)\qquad\text{for all }(v,u,q)\in X\,.$$

In particular, since $I : (X, d^*_\infty) \to L^1$ is a pointwise limit of continuous maps, it is a Baire-1 map. Therefore the set of points of continuity of I is residual in (X, d^*_∞), see [19].

Step 3 - points of continuity of the identity map

We claim that if $(v, u, q) \in X$ is a point of continuity of I, then

$$(v, u)(x, t) \in K \qquad \text{for a.e. } (x, t) \in \Omega\,.$$

By the definition of X, such (v, u, q) must be the strong L^1 limit of some sequence $\{(v_k, u_k, q_k)\} \subset X_0$.

Therefore, with $p_k = q_k - \frac{1}{n}|v_k|^2$, and

$$f_k = \operatorname{div}\left(v_k \otimes v_k - \frac{1}{n}|v_k|^2 Id - u_k\right),$$

we obtain $\operatorname{div} v_k = 0$ and

$$\partial_t v_k + \operatorname{div} v_k \otimes v_k + \nabla p_k = f_k.$$

Moreover, $f_k \to 0$ in H^{-1}. Therefore it remains to prove our claim.

Assume for a contradiction that $(v, u, q) \in X$ is a point of continuity of I and the set $\{(x, t) \in \Omega : (v, u) \notin K\}$ has positive measure. Then there exists a point $z = (\bar v, \bar u, \bar q) \in \overline{\mathcal{U}}$ with $(\bar v, \bar u) \in K^{co} \setminus K$ such that for all $\varepsilon > 0$ the set $\{(x, t) \in \Omega : |(v, u, q) - z| < \varepsilon\}$ has positive measure. Since $(\bar v, \bar u) \in K^{co} \setminus K$, by Carathéodory's theorem it can be written as a finite convex combination

$$(\bar v, \bar u) = \sum_{i=1}^{n(n+3)/2} \lambda_i (\bar v^i, \bar u^i),$$

with $(\bar v^i, \bar u^i) \in K$, and where at least two λ's are different from zero, say λ_1, λ_2. Let $\tau_0 = \frac{1}{2}\min\{\lambda_1, 1 - \lambda_1, \lambda_2, 1 - \lambda_2\}$ and consider the line segment

$$\sigma = \left\{\tau(\bar v^1 - \bar v^2, \bar u^1 - \bar u^2, 0) : -\tau_0 \le \tau \le \tau_0\right\}. \tag{24}$$

The choice of τ_0 guarantees that $z + \sigma \subset \overline{\mathcal{U}}$, where $z + \sigma = \{z + w : w \in \sigma\}$. Furthermore there exists $\varepsilon > 0$ such that

$$\tilde z \in \mathcal{U} \text{ with } |z - \tilde z| < \varepsilon \implies \tilde z + \sigma \subset \mathcal{U}. \tag{25}$$

With this choice of ε let

$$m = \left|\{(x, t) \in \Omega : |(v, u, q) - z| < \varepsilon/2\}\right| > 0.$$

By the definition of X there exists a sequence $(v^k, u^k, q^k) \in X_0$ with

$$(v^k, u^k, q^k) \overset{*}{\rightharpoonup} (v, u, q) \text{ in } L^\infty$$

and since (v, u, q) is a point of continuity of I,

$$(v^k, u^k, q^k) \to (v, u, q) \text{ in } L^1.$$

In particular we may assume that

$$\left|\left\{(x,t)\in\Omega:\ |(v^k,u^k,q^k)-z|<\varepsilon\right\}\right|\geq\frac{1}{2}m.$$

Since (v^k,u^k,q^k) are smooth, for each k there exists an open subset $\Omega_k\subset\Omega$ and $\tilde\varepsilon_k\in(0,\varepsilon)$ such that $|\Omega_k|\geq\frac{1}{4}m$,

$$|(v^k,u^k,q^k)-z|<\varepsilon \text{ for all } (x,t)\in\Omega_k, \tag{26}$$

and

$$\operatorname{dist}\Big((v^k,u^k,q^k),\partial\mathcal{U}\Big)>\tilde\varepsilon_k \text{ for all } (x,t)\in\Omega_k. \tag{27}$$

Let us fix $k\in\mathbb{N}$ for a moment. For any $\tilde z\in\mathcal{U}$ satisfying $|\tilde z-z|<\varepsilon$ and $\operatorname{dist}(\tilde z,\partial\mathcal{U})>\tilde\varepsilon_k$ the line segment $\tilde z+\sigma$ is contained in $\mathcal{U}$. Therefore by compactness there exists $\delta_k>0$ such that, denoting by $\mathcal{N}_{\delta_k}(\tilde z+\sigma)$ the δ_k-neighborhood of the line segment $\tilde z+\sigma$, we have

$$\mathcal{N}_{\delta_k}(\tilde z+\sigma)\subset\mathcal{U} \text{ for all } \tilde z \text{ satisfying } |\tilde z-z|<\varepsilon \text{ and } \operatorname{dist}(\tilde z,\partial\mathcal{U})>\tilde\varepsilon_k. \tag{28}$$

Note that the calculation at the end of Step 1 shows that σ defined in (24) satisfies the assumptions in Proposition 2.2. Therefore, for each fixed $k\in\mathbb{N}$ the proposition yields a smooth solution $(\hat v^k,\hat u^k,\hat q^k)$ to (8) such that $\operatorname{supp}(\hat v^k,\hat u^k,\hat q^k)\subset B_1(0)$,

$$(\hat v^k,\hat u^k,\hat q^k)\subset\mathcal{N}_{\delta_k/2}(\sigma) \text{ for all } (x,t)\in\mathbb{R}^n_x\times\mathbb{R}_t, \tag{29}$$

$$\int|\hat v^k(x,t)|dxdt\geq\alpha. \tag{30}$$

By an additional rescaling argument we can build a new triple $(\check v^k,\check u^k,\check q^k)$ with the properties that

$$(\check v^k,\check u^k,\check q^k)\subset\mathcal{N}_{\delta_k/2}(\sigma) \text{ for all } (x,t)\in\mathbb{R}^n_x\times\mathbb{R}_t, \tag{31}$$

$$d^*_\infty\big((\check v^k,\check u^k,\check q^k),0\big)<1/k, \tag{32}$$

$$\operatorname{supp}(\check v^k,\check u^k,\check q^k)\subset\Omega_k, \tag{33}$$

and

$$\int|\check v^k(x,t)|dxdt\geq\frac{1}{2}\alpha|\Omega_k|\geq\frac{1}{8}\alpha m. \tag{34}$$

Indeed, for every radius $r>0$ and every $\xi\in\mathbb{R}^{n+1}$, consider the rescaled map

$$(\hat v^k,\hat u^k,\hat q^k)_{\xi,r}(x,t)\ =\ (\hat v^k,\hat u^k,\hat q^k)\left(\frac{(x,t)-\xi}{r}\right).$$

For each $\rho>0$ by a well–known covering argument, we can find a finite family of pairwise disjoint balls $B_{r_i}(\xi_i)$ contained in Ω_k, such that

$$\Big|\Omega_k\setminus\bigcup_i B_{r_i}(\xi_i)\Big|\leq\frac{\alpha}{2}|\Omega_k|$$

and $r_i \leq \rho$. Hence, if we set

$$(\check{v}^k, \check{u}^k, \check{q}^k)(x,t) := \begin{cases} (\hat{v}^k, \hat{u}^k, \hat{q}^k)_{\xi_i, r_i}(x,t) & \text{if } (x,t) \in B_{r_i}(\xi_i) \\ \\ 0 & \text{otherwise,} \end{cases} \tag{35}$$

this new map satisfies the properties (31), (33) and (34). Now, choosing ρ small, we get $(\check{v}^k, \check{u}^k, \check{q}^k)$ arbitrarily close to 0 in the weak* topology of L^∞, and hence we satisfy the requirement (32). Note that here $\alpha > 0$ depends on σ, but not on $k \in \mathbb{N}$.

Next, consider the sequence $(\tilde{v}^k, \tilde{u}^k, \tilde{q}^k)$ defined by

$$(\tilde{v}^k, \tilde{u}^k, \tilde{q}^k) = (v^k, u^k, q^k) + (\check{v}^k, \check{u}^k, \check{q}^k).$$

We claim first of all that $(\tilde{v}^k, \tilde{u}^k, \tilde{q}^k) \in X_0$. Indeed, it is smooth, solves (8) and $\operatorname{supp}(\tilde{v}^k, \tilde{u}^k, \tilde{q}^k) \subset \Omega$. Furthermore,

$$(\tilde{v}^k, \tilde{u}^k, \tilde{q}^k) = (v^k, u^k, q^k) \in \mathcal{U} \quad \text{ for } (x,t) \notin \Omega_k.$$

Finally, if $(x,t) \in \Omega_k$ then $\tilde{z} = (v^k(x,t), u^k(x,t), q^k(x,t))$ satisfies (28) because of (26) and (27), hence $(\tilde{v}^k(x,t), \tilde{u}^k(x,t), \tilde{q}^k(x,t)) \in \mathcal{U}$ because of (31).

From (34) we deduce that

$$\|\tilde{v}^k - v^k\|_{L^1(\Omega)} \geq \frac{1}{8}\alpha m > 0 \text{ for all } k.$$

Since $v^k \to v$ strongly in L^1, this implies that $\tilde{v}^k$ does not converge strongly to v. On the other hand (32) implies that

$$(\check{v}^k, \check{u}^k, \check{q}^k) \overset{*}{\rightharpoonup} (0,0,0) \quad \text{ in } L^\infty,$$

i.e. that

$$(\tilde{v}^k, \tilde{u}^k, \tilde{q}^k) \to (v,u,q) \quad \text{ in } (X, d^*_\infty).$$

This contradicts the continuity of I at (v,u,q) and hence completes the proof. □

References

[1] Bressan, A., and Flores, F. On total differential inclusions. *Rend. Sem. Mat. Univ. Padova 92* (1994), 9–16.

[2] Cellina, A. On the differential inclusion $x' \in [-1, +1]$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 69*, 1-2 (1980), 1–6 (1981).

[3] Chorin, A. J. *Vorticity and turbulence*, vol. 103 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[4] Constantin, P., E, W., and Titi, E. S. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys. 165*, 1 (1994), 207–209.

[5] Dacorogna, B., and Marcellini, P. General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. *Acta Math. 178* (1997), 1–37.

[6] Dafermos, C. M. *Hyperbolic conservation laws in continuum physics*, vol. 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[7] DiPerna, R. J. Compensated compactness and general systems of conservation laws. *Trans. Amer. Math. Soc. 292*, 2 (1985), 383–420.

[8] DiPerna, R. J., and Majda, A. J. Concentrations in regularizations for 2-D incompressible flow. *Comm. Pure Appl. Math. 40*, 3 (1987), 301–345.

[9] Duchon, J., and Robert, R. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations *Nonlinearity*, 13 (2000), 249–255.

[10] Eyink, G. L. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. *Phys. D 78*, 3-4 (1994), 222–240.

[11] Frisch, U. *Turbulence*. Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.

[12] Gromov, M. *Partial differential relations*, vol. 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1986.

[13] Kirchheim, B. Deformations with finitely many gradients and stability of quasiconvex hulls. *C. R. Acad. Sci. Paris Sér. I Math. 332*, 3 (2001), 289–294.

[14] Kirchheim, B. Rigidity and Geometry of microstructures. Habilitation thesis, University of Leipzig, 2003.

[15] Kirchheim, B., Müller, S., and Šverák, V. Studying nonlinear PDE by geometry in matrix space. In *Geometric analysis and Nonlinear partial differential equations*, S. Hildebrandt and H. Karcher, Eds. Springer-Verlag, 2003, pp. 347–395.

[16] Majda, A. J., and Bertozzi, A. L. *Vorticity and incompressible flow*, vol. 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[17] Müller, S., and Šverák, V. Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. of Math. (2) 157*, 3 (2003), 715–742.

[18] Onsager, L. Statistical hydrodynamics. *Nuovo Cimento (9) 6*, Supplemento, 2(Convegno Internazionale di Meccanica Statistica) (1949), 279–287.

[19] Oxtoby, J. C. *Measure and category*, second ed., vol. 2 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980. A survey of the analogies between topological and measure spaces.

[20] Scheffer, V. An inviscid flow with compact support in space-time. *J. Geom. Anal. 3*, 4 (1993), 343–401.

[21] Shnirelman, A. On the nonuniqueness of weak solution of the Euler equation. *Comm. Pure Appl. Math. 50*, 12 (1997), 1261–1286.

[22] Shnirelman, A. Weak solutions with decreasing energy of incompressible Euler equations. *Comm. Math. Phys. 210*, 3 (2000), 541–603.

[23] Tao, T. *Nonlinear dispersive equations*, vol. 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.

[24] Tartar, L. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, vol. 39 of *Res. Notes in Math.* Pitman, Boston, Mass., 1979, pp. 136–212.

[25] Tartar, L. The compensated compactness method applied to systems of conservation laws. In *Systems of nonlinear partial differential equations (Oxford, 1982)*, vol. 111 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* Reidel, Dordrecht, 1983, pp. 263–285.

[26] Temam, R. *Navier-Stokes equations*, third ed., vol. 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1984. Theory and numerical analysis, With an appendix by F. Thomasset.

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